From the resonance theory to the statistical model

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We present a brief introduction to the R-matrix theory for the compound nuclear reaction, and explain its relation to the energy-average cross section in the unresolved resonance region. To bridge these two representations of the compound reaction cross section, we utilize the Gaussian Orthogonal Ensemble (GOE) embedded in the scattering matrix to calculate the average cross sections, and demonstrate calculation of the decay width from a transmission coefficient that is a model input.

1. Introduction

When a slow neutron interacts with a nucleus, the reaction cross section shows a distinct resonating structure, in which each of the peak location corresponds to an eigenstate of the compound system just above the neutron separation energy. In the case of a fast-energy neutron (in the keV to MeV region), the reaction cross section no longer shows sharp resonances because the compound states strongly overlap each other. Under this circumstance, each of the resonances cannot be resolved, and an energy-average cross section is only meaningful. The average cross section can be related to statistical properties of resolved resonances, namely the average resonance spacing D and decay widths $\langle\Gamma\rangle$. These statistical properties are often studied by applying the Gaussian Orthogonal Ensemble (GOE) implemented in the S and K-matrices. Although this is an old problem, our recent development on the GOE Monte Carlo technique [1, 2] sheds a new light on some long standing compound nuclear reaction problems.

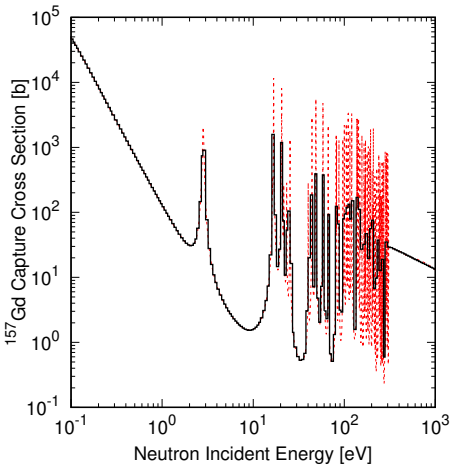

Fig. 1: Evaluated ^{157}Gd capture cross section from the thermal to the fast energy. The thin dotted curve is for the 0 K cross section reconstructed from the resolved resonance parameters. The thick histogram is a grouped cross section in the SAND-II 640 energy group structure.

Figure 1 shows the neutron radiative capture cross section of ^{157}Gd in JENDL-4.0 at low energies, where the fluctuation suddenly disappears beyond 300 eV. This is because the representation of cross section changes at this boundary energy. Above

300 eV, the cross sections are energy-averaged, while they are not blow that energy. This can be seen when we calculate a group cross section shown by the thick histogram; the cross section shape in the 100 - 300 eV region is converging to the smooth straight line at higher energies.

2. Compound Nuclear Reaction

2.1 R-matrix theory

We still base an interpretation of compound nuclear reaction upon Bohr's Hypothesis; (i) an incident particle shares its energy with the target nucleons, (ii) a compound nucleus (CN) attains statistical equilibrium, and (iii) the decay modes of CN are independent of formation. Several theories were formulated to characterize the resonances in CN, which are by Breit and Wigner [3], Kapur and Peierls [4], and the most commonly quoted one is the R-matrix theory of Wigner and Eisenbud [5]. R-matrix is defined by the energy of λ-th resonance E_λ, and the decay amplitude $\gamma_{\lambda c}$ into the channel c,

$$R_{cc'} = \sum_\lambda \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E} \ , \qquad (1)$$

which yields the scattering matrix S [6]. Instead of $\gamma_{\lambda c}$, the decay width $\Gamma_{\lambda c} = 2P_c\gamma_{\lambda c}^2$ is often used, where P_c is a penetration factor [6]. These parameters are determined by fitting calculated quantities to available experimental data, and in this sense the R-matrix formula is not a predictive theory, nevertheless no approximation was made to derive the equation. The resonance energy E_λ can be estimated from the nuclear excited state in CN when the nuclear structure is known. In Fig. 2, resonances for neutron-induced reaction on ^{12}C are depicted on the right, and the excited state in ^{13}C corresponding to each resonance is shown on the left. The first resonance occurs at 2.077 MeV, while the 6.864 MeV $(5/2)^+$ level corresponds to this resonance.

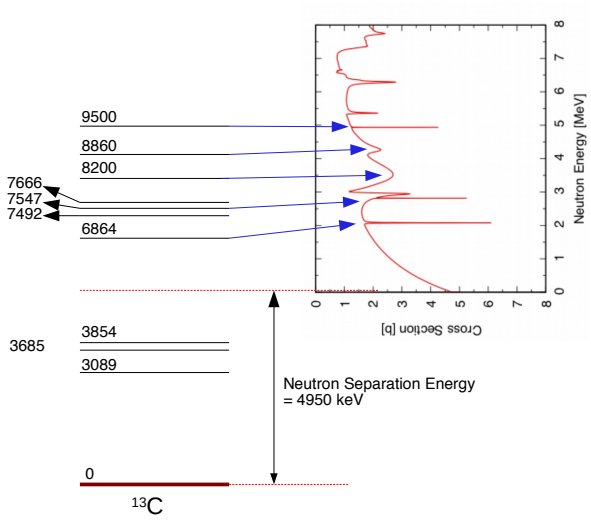

Fig. 2: The ^{12}C total cross section is shown on the right side, and the left side is nuclear structure of ^{13}C and some of the excited states that correspond to the resonances.

One of the significant features of the R-matrix theory is the S-matrix unitarity, $S^\dagger S = 1$, which ensures the flux conservation. Because of this one can estimate some experimentally unknown cross section by inverse reactions, *e.g.*, prediction of (n, α) by (α, n) data.

2.2 Approximations to R-matrix theory

Although the R-matrix theory is exact, a downside is that it requires enough experimental data to determine all the channels. This situation becomes more severe when the radiative capture

channel has a lot of final states. This is one of the main reasons why the full R-matrix analysis is limited to light elements.

In the case of resonance reactions on medium to heavy nuclei, there are many small photon channels $\gamma_{\lambda c}$, and their sign is random. When all the photon emission channels are lumped into a single capture channel Γ_γ, as an approximation often made in the single-level (SLBW) or multi-level Breit-Wigner (MLBW) formulae [7], interference between resonances will have some issues. The Reich-Moore (RM) approximation [8], which eliminates all the photon emission channels and lumps them into one capture channel, has better treatment of resonance interference. The RM R-matrix reads

$$R_{cc'}^{\mathrm{RM}} = \sum_{\lambda} \frac{\gamma_c \gamma_{c'}}{E_\lambda - E - i\Gamma_\gamma/2} \ . \qquad (2)$$

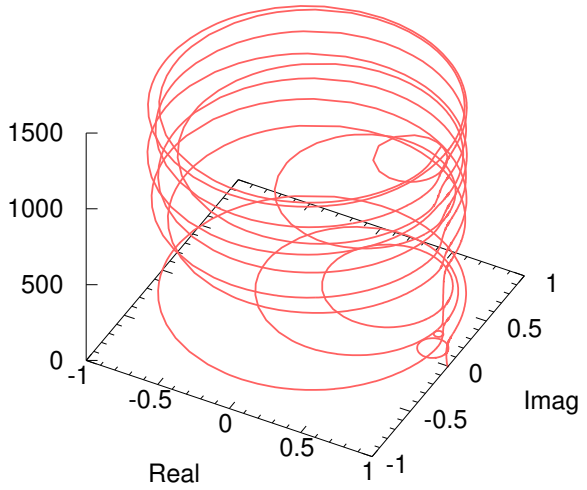

Fig. 3: S-matrix elements (elastic channel) on the complex plane for neutron induced reaction on ^{184}W with the Reich-Moore approximation. The vertical axis is the neutron incident energy in eV.

When these approximations are applied, the S-matrix is no longer unitary, and the flux deficit depends on the size of capture width. Figure 3 demonstrates the S-matrix elements of ^{184}W for the elastic channel calculated from the RM resonance parameters in JENDL-4. The incident neutron energy changes from zero at the bottom plane to 1.5 keV at the top. At each resonance, the S-matrix draws a circle in the complex plane, and smaller circles correspond to the large capture cross section case, where the unitary deficit is also large.

Although RM and MLBW S-matrices are not unitary, one can still calculate angular distributions of elastically scattered neutrons by applying the Blatt-Biedenharn formula [9]. This is particularly important for nuclides near the magic numbers, where non-statistical behavior of the resonance distribution sometimes enhances the neutron scattering in the forward and backward angles. Such scattering angular distributions are very different from predictions by the optical model that often gives a very small p-wave contribution in the resonance range [10]. Neutron leakage is enhanced when more neutrons are scattered in the forward angles, while larger backward scattering increases an effective neutron multiplicity k_{eff}.

2.3 Random matrix approach in unresolved resonance region

Since individual resonances cannot be resolved anymore above the resolved resonance region, energy-average cross sections and/or average resonance parameters ($\langle\Gamma_c\rangle$ and D) are given in evaluated nuclear data files. The channel degree-of-freedom ν_c of the χ^2-distribution for Γ_c also characterizes the distribution of cross sections. When $\nu_c = 1$, the width distribution is reduced to the well-know Porter-Thomas distribution. The resonance spacing is also known to

form the Wigner distribution. Sometimes it is claimed that these distributions are obtained by applying GOE to the compound nuclear reaction. However, they are not directly connected but inspired [7]. A realistic implementation of GOE in the scattering matrix S was first proposed by Verbaarschot *et al.* [11], and ensemble average of the calculated cross sections was performed by the Grassmann integration technique.

The energy-average compound reaction cross section from channel a to b can be defined by the average decay width

$$\sigma_{ab}^{\rm CN} = \frac{2\pi}{D}\left\langle \frac{\Gamma_a\Gamma_b}{\sum_c \Gamma_c}\right\rangle = \frac{2\pi}{D}\frac{\langle\Gamma_a\rangle\langle\Gamma_b\rangle}{\sum_c\langle\Gamma_c\rangle}W_{ab} = \frac{T_aT_b}{\sum_c T_c}W_{ab}\ , \tag{3}$$

where W_{ab} is the width fluctuation correction factor [12], and T_c is the particle transmission coefficient. In Eq. (3), a weak-coupling limit approximation $T_c \simeq 2\pi\langle\Gamma_c\rangle/D$ is used. This was examined in the strong-coupling regime [13] by applying the GOE model [1].

Defining the compound cross section by Γ_c is somewhat ambiguous, nevertheless the statistical model has been developed in such a way. In fact the partial decay amplitude γ_c in the GOE model can be defined by two matrices, S and K. When the random matrix $H^{\rm GOE}$ is first diagonalized, the K-matrix that has a similar form to R includes the partial decay amplitude in the numerator,

$$K = \pi W^T\frac{1}{E-H^{\rm GOE}}W, \qquad K_{cc'} = \frac{1}{2}\sum_\lambda \frac{\tilde\gamma_{\lambda c}\tilde\gamma_{\lambda c'}}{E-E_\lambda}\ , \tag{4}$$

where W is the coupling matrix [1]. The average width $\langle\Gamma_c\rangle$ can be evaluated by an ensemble average of $\tilde\gamma_{\lambda c}^2$. On the other hand, a pole-expansion form of the S-matrix gives another decay width [6].

$$S_{cc'} = \delta_{cc'} - i\sum_\nu \frac{\gamma_{\nu c}\gamma_{\nu c'}^*}{E-E_\nu+i\Gamma_\nu/2}\ . \tag{5}$$

The top panels in Fig. 4 show an example of GOE realizations for a two-channel case (elastic and inelastic scattering only). The left panel is for the weak-coupling case ($T_c = 0.1$), and the right panel is for the strong-coupling case ($T_c = 0.9$). The bottom panels are the normalized average width $\pi\langle\Gamma_a\rangle/D$ plotted at each resonance energy E_λ. In the small T_a case, both the K and S-matrices give very similar decay width, while they differ in the strong absorption case. This situation becomes more complicated when we calculate $\langle\Gamma_a\Gamma_b/\sum\Gamma_c\rangle$ or $\langle\Gamma_a\rangle\langle\Gamma_b\rangle/\sum\langle\Gamma_c\rangle$ by the GOE model, hence definition of the cross section by the decay width would be open to argument. We are investigating the equality in Eq. (3) by performing the Monte Carlo calculation for $\langle\Gamma_a\Gamma_b/\sum\Gamma_c\rangle$ and $\langle\Gamma_a\rangle\langle\Gamma_b\rangle/\sum\langle\Gamma_c\rangle$. However, so far what we can conclude is that the last term of $T_aT_b/\sum T_cW_{ab}$ is the most accurate expression for the compound nuclear reaction.

3. Conclusion

A brief introduction to the R-matrix theory for the compound nuclear reaction is given, and its relation to the energy-average cross section in the unresolved resonance region was summarized.

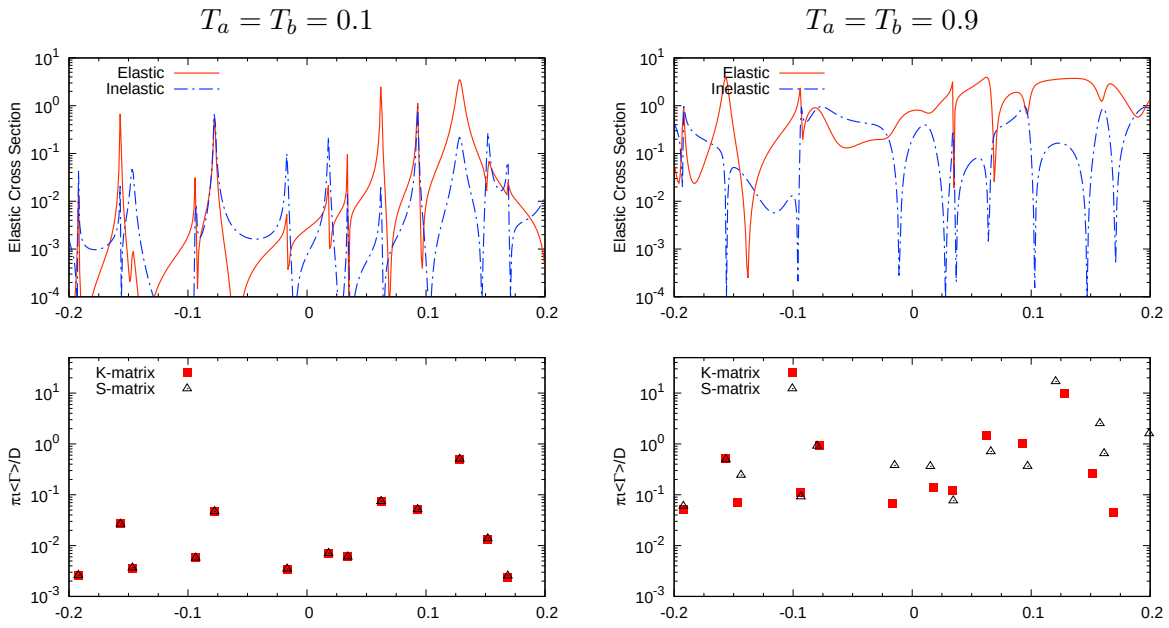

Fig. 4: An example of GOE model realization. The top panels are the calculated cross sections for the weak and strong coupling cases, where the same random number sequence was used. The bottom panels are the calculated resonance decay width by K and S matrices.

We utilized the Gaussian Orthogonal Ensemble (GOE), which is embedded in the scattering matrix, to calculate the energy-average cross section by a given transmission coefficient. It was shown that defining compound reaction cross sections in terms of the decay width might be somewhat ambiguous.

Acknowledgment

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