Closed orbit distortion in scaling FFAGs

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COD – the standard picture

Assuming a linear lattice, the equation of motion is given by Hill's equation.

$$x'' + k_x x = \theta_x \qquad\qquad y'' + k_y y = \theta_y$$

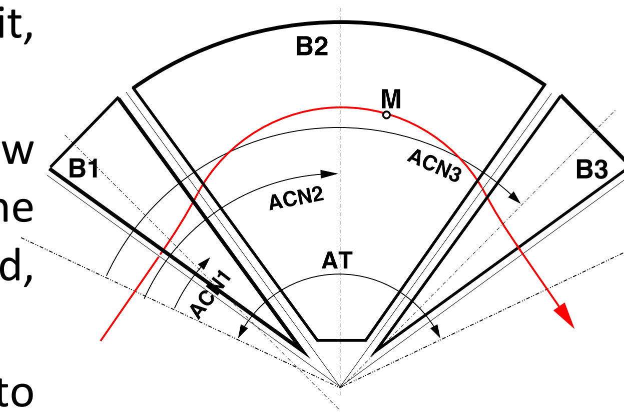
Here we assume motion around a reference orbit, $x=x_0$,$y=y_0$ when θ=0. Imposing the closed orbit condition x(s) = x(s+C), x'(s)=x'(s+C) leads to the equation for the closed orbit response to distributed set of dipole kicks

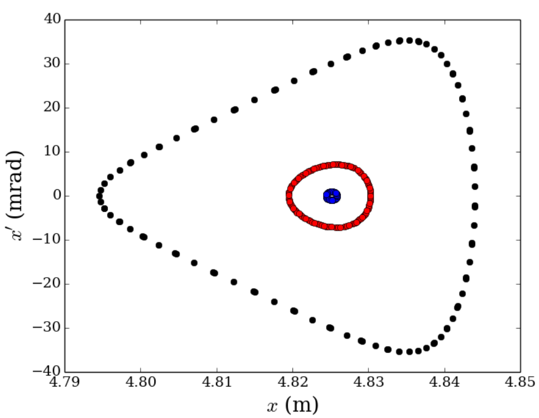
$$x_i = \sum_j \theta_j \frac{\sqrt{\beta_i \beta_j}}{2\sin(\pi q)} \cos\left(|\psi_i - \psi_j| - \pi q\right) \quad \rightarrow \quad x_i = \sum_j R_{ij}\theta_j$$

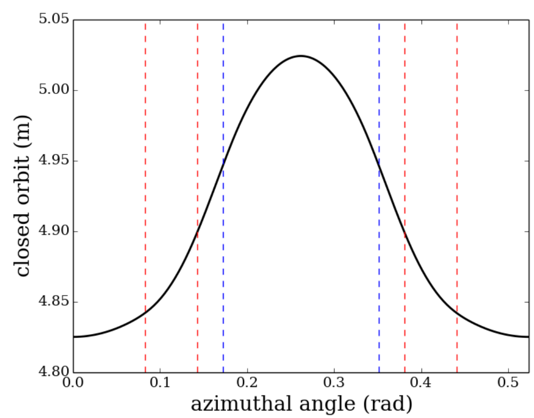
- The COD caused by set of dipole kicks is given by the orbit response matrix (RM).
- In the case of a single kick, the COD increases linearly with θ.
- The COD amplitude varies with 1/sin(πq), tending to infinity as q approaches integer.

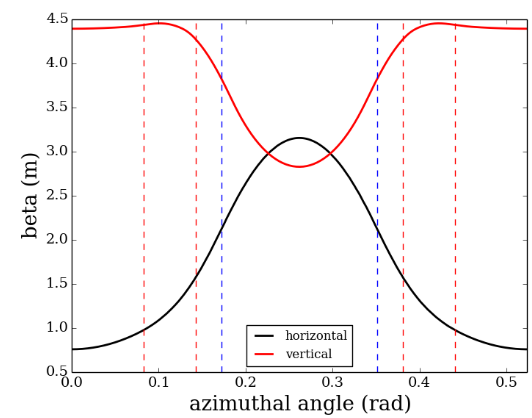
Simulation setup – bare lattice

- The analytic "FFAG" element in Zgoubi is used in this study. The PyZgoubi interface is used to find the closed orbit, calculate the optics etc.
- To find the closed orbit, track a single particle for a few turns. Record turn-by-turn x and x' at some point. If the enclosed phase space area is greater than some threshold, track again starting from the phase space centre.
- Finally, track particles with a small betatron amplitude to get the transfer matrix and hence the tune and optics.



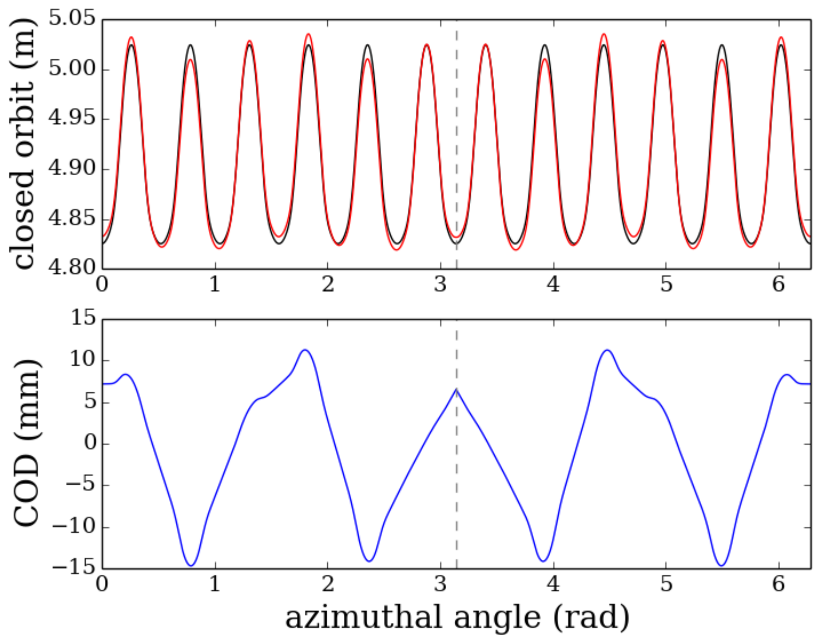






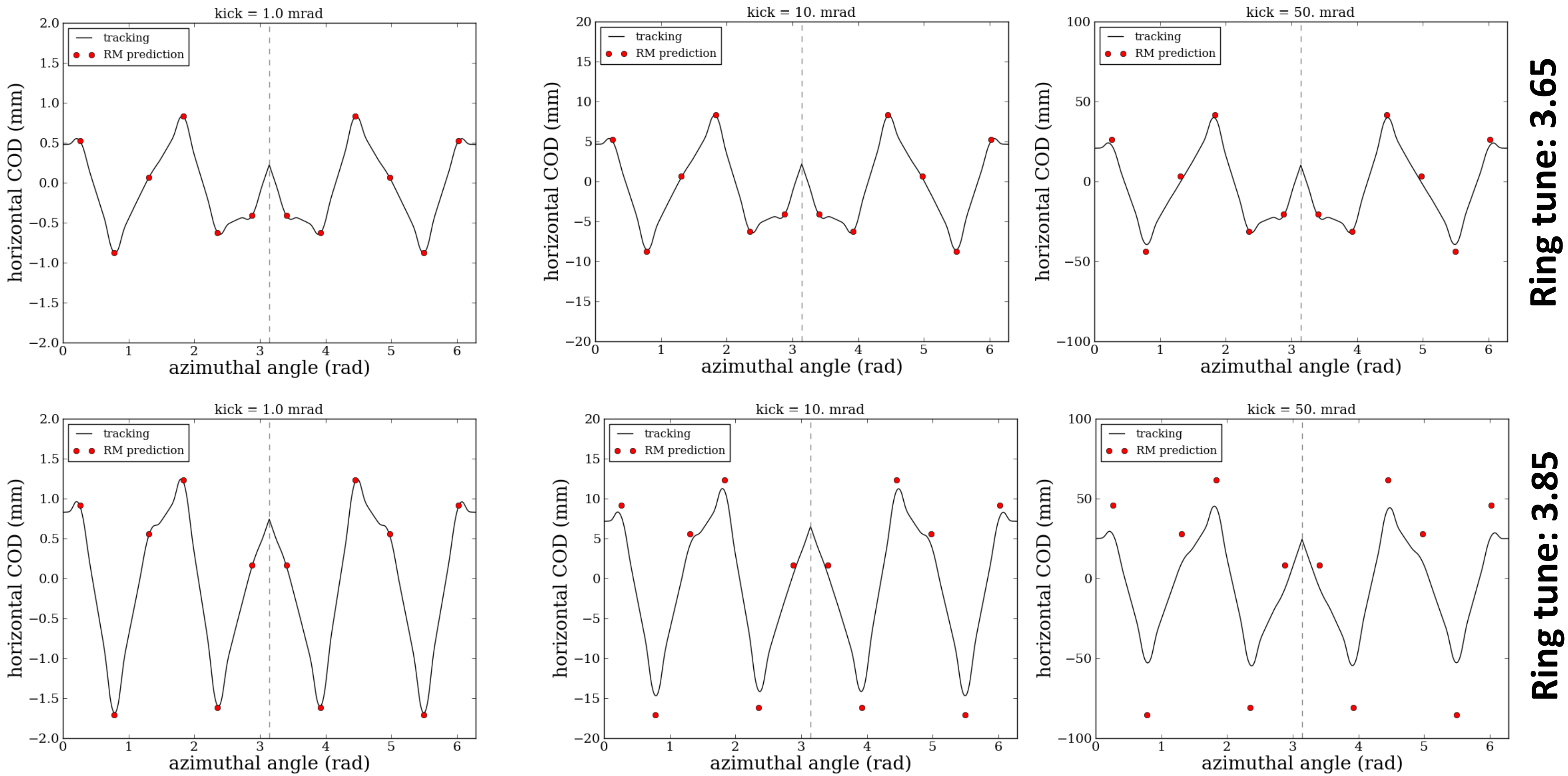
Simulation setup – error source

- Introduce a single error source. Since this breaks the symmetry, the entire ring circumference now needs to be tracked to find the closed orbit.
- The difference of the closed orbit with and without the error source is the COD.



10 mrad dipole kick

Comparison of tracking with RM prediction



COD including nonlinear components

Including nonlinearities, the equation of motion around the reference orbit is given by

$$x'' + k_x x = -\mathrm{Re}\left[\sum_{n\geq 2} \frac{k_n + ij_n}{n!}(x+iy)^n\right] + \theta_x \qquad y'' + k_y y = \mathrm{Im}\left[\sum_{n\geq 2} \frac{k_n + ij_n}{n!}(x+iy)^n\right] + \theta_y$$

Where the normal and skew gradients are $k_n(s) \equiv \frac{1}{B_0\rho_0}\frac{\delta^n B_y}{\delta x^n}$ $\qquad j_n(s) \equiv \frac{1}{B_0\rho_0}\frac{\delta^n B_x}{\delta x^n}$

- Given a finite dipole kick, the solution involves dipole feed down from all the high order components (sextupole is the leading order).
- Similarly, quadrupole feed down results in variation of the betatron tune with COD amplitude. For perturbed gradient k, detuning to first order is given by

$$\Delta Q_x = -\frac{1}{4\pi}\int_0^C \beta_x(s)\hat{k}(s)ds \qquad \Delta Q_y = -\frac{1}{4\pi}\int_0^C \beta_y(s)\hat{k}(s)ds$$

Simplified equation of motion

- The normal gradients can be expressed in term of the scaling index κ

$$B = B_0 \left(\frac{r}{r_0}\right)^{\kappa} \qquad k_n = \frac{1}{B\rho}\frac{d^n B}{dx^n} = \frac{\kappa!}{\rho r^n (\kappa - n)!}$$

- Assuming zero vertical motion and considering normal components only

$$x'' + k_x x = -\sum_{n \geq 2} \frac{\kappa!}{\rho r^n n! (\kappa - n)!} x^n + \theta_x$$

- Keeping just the leading order term (sextupole) one has

$$x'' + k_x x = -k_2 x^2 + \theta_x$$

$$x'' + k_x x = -\frac{\kappa(\kappa - 1)}{2\rho r^2} x^2 + \theta_x$$

Approximate solution

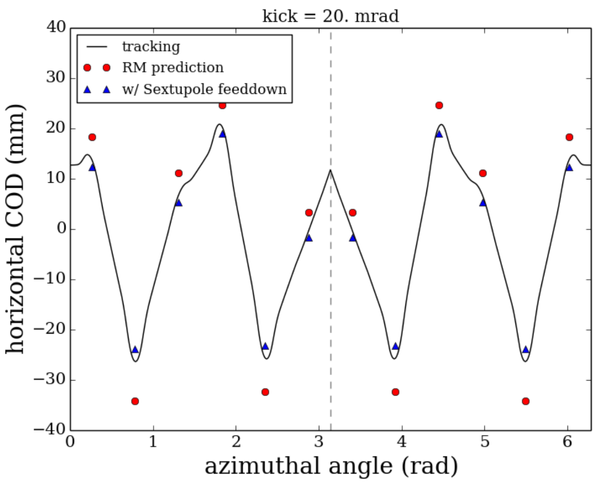
- Try an ad-hoc perturbation approach. In the first step solve the linear equation

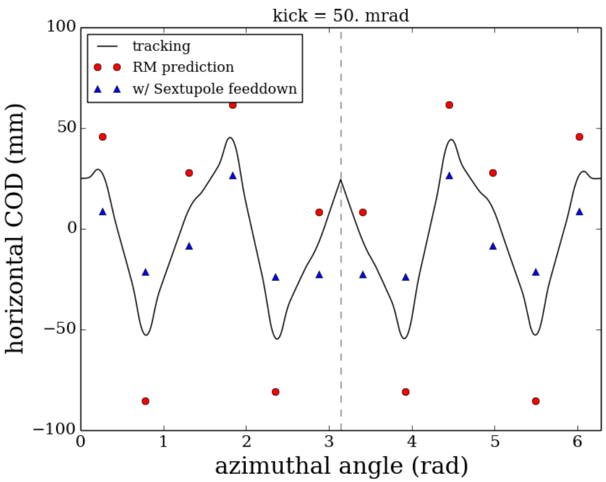
$$x'' + k_x x = \theta_x \quad \rightarrow \quad x_0(s)$$

- In the second step, substitute x_0 into the sextupole term reducing the problem to a linear one. Solve again.

$$x'' + k_x x = -\frac{\kappa(\kappa-1)}{2\rho r^2} x_0^2 + \theta_x$$

Sextupole term acts as a pseudo-kick.

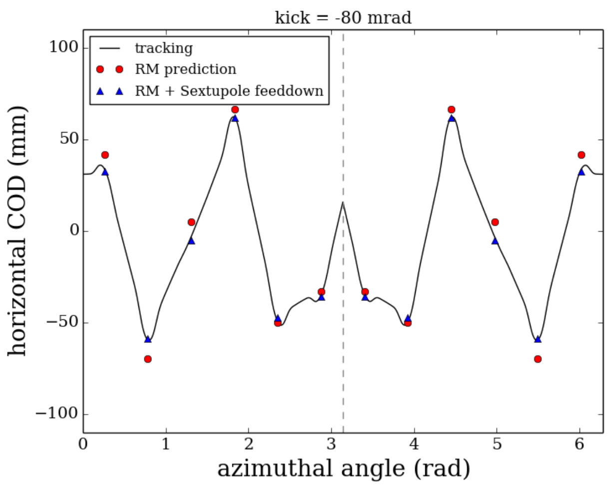


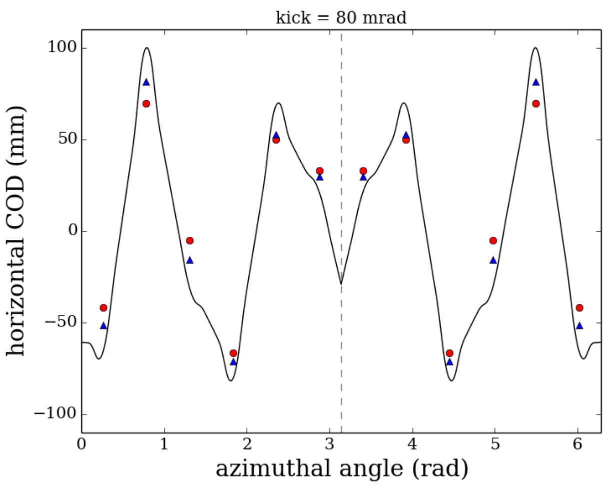


Ring tune: 3.85

Dipole kick polarity

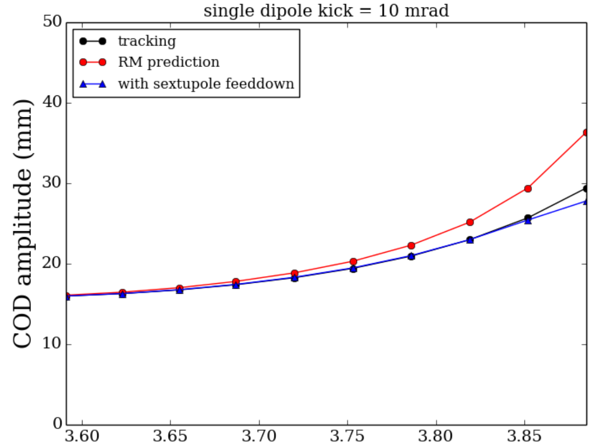
- Final COD amplitude can be greater than or less than COD predicted by RM depending on the pattern of latter.
- In the feeddown approximation, pseudo-kick produced by each sextupole should be the same even though pattern polarity of dipole is reversed.

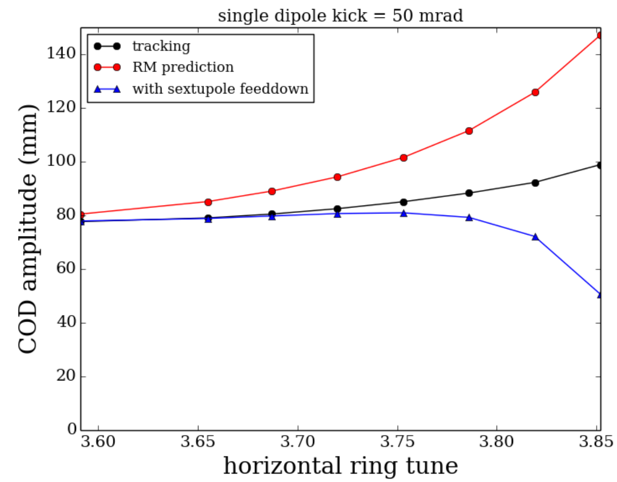


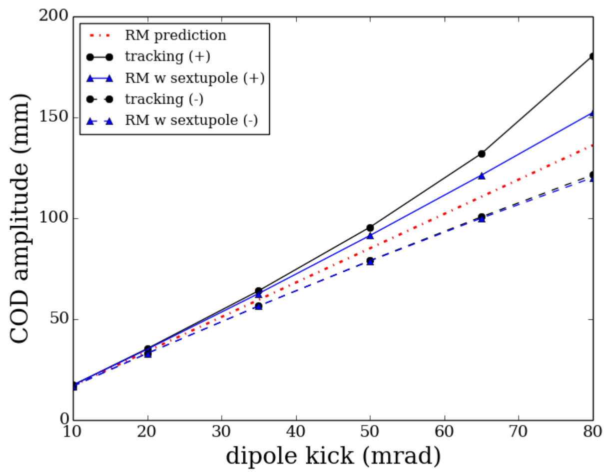


Limitations of approximation

- Sextupole feeddown approximation works well as long as the kicks are and small the the tune isn't too close to integer.





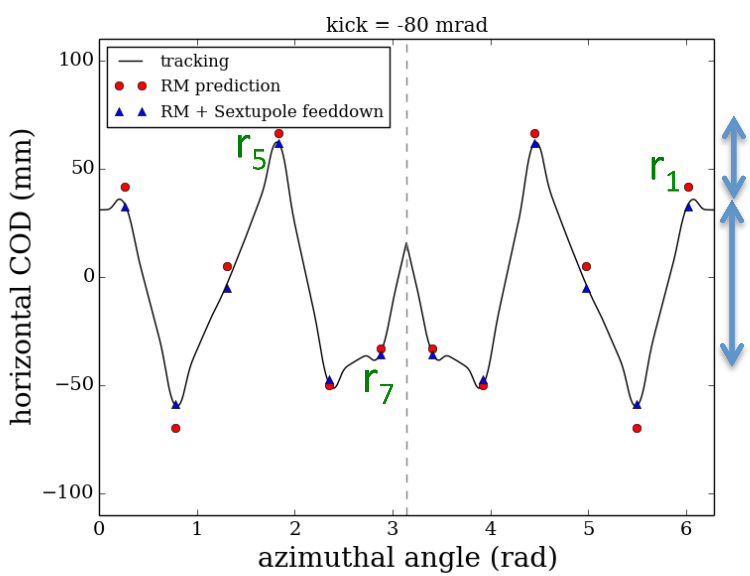


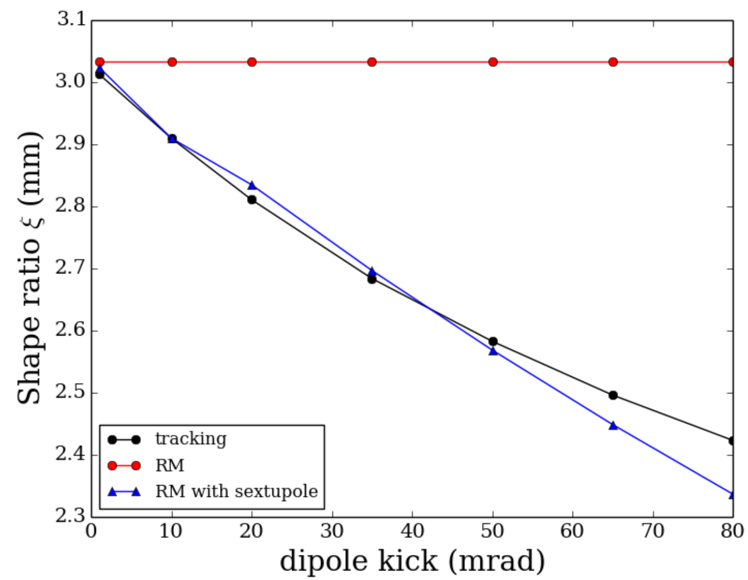
- Can see onset of nonlinear growth of COD with dipole kick amplitude.
- This occurs more strongly with one kick polarity than the other.

COD shape

- Given a single error source, the shape of the COD is independent of kick amplitude in a linear lattice. This is not true when nonlinearities are taken into account.
- Parameterise shape in terms of the ratio of the difference between closed orbits.

$$\xi = \frac{r_1(p) - r_7(p)}{r_5(p) - r_1(p)} \qquad \xi = \frac{\cos\left(2\pi q_x \Delta n_1/n - \pi q_x\right) - \cos\left(2\pi q_x \Delta n_7/n - \pi q_x\right)}{\cos\left(2\pi q_x \Delta n_5/n - \pi q_x\right) - \cos\left(2\pi q_x \Delta n_1/n - \pi q_x\right)}$$





Conclusions

- The nonlinear multipole components in the magnetic field of a scaling FFAG has an effect on the COD (both shape and amplitude). To first order, it can be considered a sextupole feeddown.
- It should be noted that the effect should be negligible if the operating point is sufficiently far from an integer tune.
- Others have studied nonlinear dynamics using Hamiltonian perturbation theory (e.g. R. Ruth). Develop to predict the effect of nonlinearities on the closed orbit in a scaling FFAG.

Bibliography

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2. G. Franchetti, A. Parfenova, I. Hofmann, "Measuring localized nonlinear components in a circular accelerator with a nonlinear tune response matrix", Phys. Rev. ST Accel Beams **11**, 094001 (2008)
3. R. Ruth, "Single particle dynamics and nonlinear resonances in circular accelerators", SLAC-PUB-3836 (1985)